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CITATION:

Tsutsui, Yohei. Sparse bounds for local smoothing operators (Harmonic Analysis and Nonlinear Partial Differential Equations). 数理解析研究所講究録別冊 2019, B74: 127-137

ISSUE DATE:

2019-04

URL:

http://hdl.handle.net/2433/244764

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**B74** (2019), 127–137

# Sparse bounds for local smoothing operators

By

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## Abstract

A sparse bound for an operator involving time integral of the wave propagator is established. The operator is concerned with maximal Riesz means and an operator in local smoothing conjecture by Sogge [19].

## § 1. Introduction and result

The sparse bound for an operator $T$ is the inequality of the form:

$$|\langle Tf, g\rangle| \lesssim \Lambda_{\mathcal{S};r,s}(f,g) := \sum_{Q\in\mathcal{S}} |Q|\langle f\rangle_{Q,r}\langle g\rangle_{Q,s}, \tag{1.1}$$

where $\mathcal{S}$ is a sparse family of cubes in $\mathbb{R}^n$ and $\langle f\rangle_{Q,r} := \left(\frac{1}{|Q|}\int_Q |f|^r dx\right)^{1/r}$, in particular $\langle f\rangle_Q := \langle f\rangle_{Q,1}$. Definition of the sparse family will be given later. There are literatures on the sparse bounds for several operators; nonintegral singular operators by Bernicot, Frey and Petermichl [3], Bochner-Riesz means by Benea, Bernicot and Luque [2] and Lacey, Mena and Reguera [16] and its square function by Carro and Domingo-Salazar[6], pseudodifferential operators in Hörmander symbol class by Beltran and Cladek [1], singular integrals and its maximal truncation by Lacey [13], spherical maximal functions by Lacey [14], etc... More general sparse bounds are discussed by Fackler and Hytönen [9].

The aim of this note is to establish the sparse bound for $T = T_\ell$ with $r = 1$ and $s = 2$; where $T_\ell f(x) := \sup_{1\le R\le 2} |T_{\ell,R} f(x)|$ and

$$T_{\ell,R} f(x) := \int_{\mathbb{R}} e^{-itR}\frac{\rho(t/2^\ell)}{t^{1+\delta}} e^{it\sqrt{-\Delta}}\varphi(D)f(x)dt.$$

Received July 23, 2018. Revised January 31, 2019.
2010 Mathematics Subject Classification(s): 42B25, 42B37
*Key Words*: *sparse bound*, *local smoothing operator*, *Riesz means*
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Here, $\mathbb{N} \ni \ell \gg 1$, $\delta > 0$, $\rho \in C_0^\infty(\mathbb{R})$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ with supp $\rho \subset [1/4, 4]$ and supp $\varphi \subset \{\xi \in \mathbb{R}^n; 1/2 \le |\xi| \le 2\}$. Also, $\varphi(D)f := \mathcal{F}^{-1}[\varphi \hat{f}]$ and $e^{it\sqrt{-\Delta}}f(x) := \int e^{ix\xi}e^{it|\xi|}\hat{f}(\xi)d\xi$.

Let $\eta \in (0,1)$. A family $\mathcal{S}$ of cubes in $\mathbb{R}^n$ is called $\eta$-sparse family if there exist pairwise disjoint subsets $\{E_Q\}_{Q\in\mathcal{S}}$ such that $E_Q \subset Q$ and $|E_Q| \ge \eta|Q|$. Since $\eta$ is not important, we sometime ignore it. Owing to this property of $\{E_Q\}_{Q\in\mathcal{S}}$, it follows that if $1 \le r < p < s' < \infty$,

$$\Lambda_{\mathcal{S};r,s}(f,g) \lesssim \eta^{-1}\|f\|_{L^p}\|g\|_{L^{p'}},$$

which implies the $L^p$-boundedness of $T$ follows from (1.1). Here, we remark that the implicit constant is independent of the sparse family $\mathcal{S}$, although the family may depend on functions. Weak bounds are also well-known, see [7] and [11]. Moreover, we can derive sharp weighted inequalities in the scale of Muckenhoupt weights for operators fulfilling (1.1) with $r = s = 1$ through

$$\Lambda_{\mathcal{S};1,1}(f,g) \lesssim [w]_{A_p}^{\max(1,1/(p-1))}\|f\|_{L^p(w)}\|g\|_{L^{p'}(\sigma)},$$

where $\|f\|_{L^p(w)} := \|fw^{1/p}\|_{L^p}$,

$$[w]_{A_p} := \sup_Q \langle w\rangle_Q \langle\sigma\rangle_Q^{p-1} \quad \text{and} \quad \sigma := w^{1-p'}$$

for all $p \in (1,\infty)$. A simple proof of this inequality is found in Moen [18].

The motivation of our study for $T_\ell$ comes from two problems; maximal Riesz means and local smoothing conjecture. To explain the connection with the former, let us define, for $\delta > 0$ and $a > 0$,

$$m_{\delta,a}(\xi) := (1 - |\xi|^a)_+^\delta,$$

where $t_+ := \max(t,0)$. $m_{\delta,a}(D)f := \mathcal{F}^{-1}[m_{\delta,a}\hat{f}]$ is called Riesz means, in particular when $a = 2$, the operator is Bochner-Riesz means. It is conjectured that $m_{\delta,2}(D)$ is bounded in $L^p$ if and only if $\delta > \delta(p) := \max\left(n\left|\frac{1}{2} - \frac{1}{p}\right| - \frac{1}{2}, 0\right)$. When $n = 2$, the sufficiency of the condition on $\delta$ was showed by Carleson and Sjölin [5]. The maximal Riesz means is defined by

$$m_{\delta,1}^*(D)f := \sup_{R>0}|m_{\delta,1}(D/R)f|.$$

For the boundedness of the maximal operator, the condition on $\delta$ seems to be more restricted, like the case $\alpha = 2$, see Tao [21]. Using the fact that the Fourier transform of $h(s) := (-s)_+^\delta$ is $c_\delta(t + i0)^{-(1+\delta)}$ in the sense of tempered distributions on $\mathbb{R}$, with $c_\delta := i^{1+\delta}\Gamma(\delta + 1)$, we have the representation:

$$m_{\delta,1}(D/R)f(x) = (2\pi)^n c_\delta R^{-\delta}\int_{\mathbb{R}} e^{-itR}\frac{1}{(t+i0)^{1+\delta}}e^{it\sqrt{-\Delta}}f(x)dt.$$

The integral near the origin is dominated by the Hardy-Littlewood maximal function. If we dyadically decompose the rest of the integral with $\rho$, each term of the sum is of the form

$$\int e^{-itR} t^{-(1+\delta)} \rho(t/2^{\ell}) e^{it\sqrt{-\Delta}} f(x) dt \text{ with } \ell \in \mathbb{N}.$$

This equals $T_{\ell,R} f(x)$ without the frequency localization operator $\varphi(D)$. Combining this argument and Littlewood-Paley decomposition, we can see that boundedness of $m^*_{\delta,1}(D)$ can be reduced to that of $T_\ell$. For the detail, see Sogge [20].

Next, we explain the connection with local smoothing conjecture. Changing the variables, we observe that $T_\ell f(x)$ is controlled by

$$T^*_\ell g(x) := 2^{-\ell\delta} \int_1^2 \left| e^{it\sqrt{-\Delta}} \varphi_\ell(D) g(x) \right| dt$$

with $g(x) := f(2^\ell x)$. Related to this operator, it is conjectured by Sogge [19] that, for any $\varepsilon > 0$,

$$\left( \int_{\mathbb{R}^n \times (1,2)} \left| e^{it\sqrt{-\Delta}} \varphi_\ell(D) f \right|^p dxdt \right)^{1/p} \lesssim 2^{\varepsilon \ell} \|f\|_{L^p}$$

for $p \in [2, 2n/(n-1)]$. Sogge in [19] and [20] proved the case $n = 2$ and $p = 4$. It should be noticed that the critical bound, i.e. $p = 2n/(n-1)$, solves affirmatively Bochner-Riesz conjecture, and as a consequence Kakeya conjecture. After writing this article, the author obtained the sparse bound for $T^*_\ell$, which is an improvement of the result in the present paper.

Let $L^\infty_0$ be a space of all functions in $L^\infty$ having compact support. Main result of this note reads as follows.

**Theorem 1.1.** *For $f, g \in L^\infty_0$ and $\varepsilon > 0$ there exist sparse families $\{\mathcal{S}^\alpha\}_{\alpha \in \{0,1,2\}^n}$ such that*

$$|\langle T_\ell f, g \rangle| \lesssim 2^{-\ell(\delta-(n-1)/2-\varepsilon)} \sum_{\alpha \in \{0,1,2\}^n} \Lambda_{\mathcal{S}^\alpha;1,2}(f,g),$$

*where the implicit constant is independent of $\ell$, $f$ and $g$.*

*Remark.* In the right hand side, the author believes that the exponent 2, in the sparse form, should be improved to 1. If this is done, then we can derive

$$\|T_\ell f\|_{L^p(w)} \lesssim 2^{-\ell(\delta-(n-1)/2-\varepsilon)} [w]_{A_p}^{\max(1,1/(p-1))} \|f\|_{L^p(w)}$$

for all $p \in (1, \infty)$ and $w \in A_p$, (i.e. $[w]_{A_p} < \infty$). As a byproduct, if $\delta > \delta(1) = (n-1)/2$, then we can see that $L^p(w) - L^p(w)$ operator norm of $m_{\delta,1}(D)$ is controlled by the same term of $w$ above, through an extrapolation [8].

## § 2. Proof of Theorem

We borrow arguments from [16] and [20].

For simplicity, we redefine $\rho(t)$ by $\frac{\rho(t)}{t^{1+\delta}}$, and then

$$\begin{aligned} T_\ell f(x) &= 2^{-\ell\delta} \sup_{1\le R\le 2} \left| \int e^{-it2^\ell R}\rho(t) e^{it2^\ell\sqrt{-\Delta}}\varphi(D)f(x)dt \right| \\ &= 2^{-\ell\delta} \sup_{1\le R\le 2} |f * K_{\ell,R}(x)|, \end{aligned}$$

where $K_{\ell,R}(x) = \int_{\mathbb{R}} e^{-i2^\ell tR}\rho(t)\int_{\mathbb{R}^n} e^{ix\xi}e^{i2^\ell t|\xi|}\varphi(\xi)d\xi dt$.

Fix a cut-off function $\psi_0 \in C_0^\infty(\mathbb{R})$ with supp $\psi_0 \subset (-1,1)$ and $\psi_0 \equiv 1$ on $(-1/2, 1/2)$. Also, we denote $\psi(\xi) = \psi_{\ell,R}(\xi) := \psi_0\left(2^{\ell\tau}(R-|\xi|)\right)$ where $\tau := 1-\varepsilon$ with $\varepsilon \in (0,1)$. Obviously, supp $\psi \subset \left\{\xi \in \mathbb{R}^n; |R-|\xi|| \le 2^{-\ell\tau}\right\}$ and $|\text{supp }\psi| \approx 2^{-\ell\tau}$. We decompose

$$T_\ell f(x) \le T_\ell^1 f(x) + T_\ell^2 f(x),$$

where $T_\ell^j f(x) = 2^{-\ell\delta}\sup_{1\le R\le 2}|f * K_{\ell,R}^j(x)|$, $(j=1,2)$ with

$$\begin{aligned} K_{\ell,R}^1(x) &= \int_{\mathbb{R}} e^{-i2^\ell tR}\rho(t)\int_{\mathbb{R}^n} e^{ix\xi}e^{i2^\ell t|\xi|}\varphi(\xi)\psi(\xi)d\xi dt \\ K_{\ell,R}^2(x) &= \int_{\mathbb{R}} e^{-i2^\ell tR}\rho(t)\int_{\mathbb{R}^n} e^{ix\xi}e^{i2^\ell t|\xi|}\varphi(\xi)(1-\psi(\xi))d\xi dt \end{aligned}$$

We will see that $T_\ell^1$ is dominant and $T_\ell^2$ can be treated as an error.

### § 2.1. Maximal operator control of $T_\ell^2$

In this subsection, we give a sparse bound of $T_\ell^2$ with $r = s = 1$. To do so, we shall show

$$T_\ell^2 f(x) \lesssim 2^{-\ell N} Mf(x), \tag{2.1}$$

with any $N > 0$ where $M$ is the Hardy-Littlewood maximal operator. Let $\chi_E$ be the characteristic function of $E \subset \mathbb{R}^n$. To see (2.1), we write

$$K_{\ell,R}^2(x) = \int_{\mathbb{R}^n} e^{ix\xi}\varphi(\xi)(1-\psi(\xi))\hat{\rho}(2^\ell t(R-|\xi|))d\xi.$$

From

$$|(-ix)^\alpha K_{\ell,R}^2(x)| \lesssim \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\int_{\mathbb{R}^n} |\partial^{\alpha_1}\varphi(\xi)|\,|\partial^{\alpha_2}(1-\psi(\xi))|\left|\partial_\xi^{\alpha_3}\hat{\rho}(2^\ell t(R-|\xi|))\right| d\xi$$

and the fact that on the support of $\varphi(1-\phi)$,

$$\left|\partial_\xi^{\alpha_3}\hat{\rho}\left(2^\ell t\left(R-|\xi|\right)\right)\right| \lesssim 2^{|\alpha_3|\ell}2^{-\varepsilon L\ell} \quad \text{with any } L > 0,$$

one obtains the pointwise bound; for any $N, N' \in \mathbb{N}$

$$|K^2_{\ell,R}(x)| \lesssim 2^{-\ell N'} \langle x\rangle^{-N},$$

where $\langle x\rangle := (1+|x|^2)^{1/2}$, which yields the pointwise bound (2.1).

To reach the desired sparse bound of $\langle T^2_\ell f, g\rangle$ from (2.1), we consider the sparse operator:

$$\Lambda_{\mathcal{S},r} f(x) := \sum_{Q\in\mathcal{S}} \langle f\rangle_{Q,r}\chi_Q(x).$$

Using the approximation property of $\mathscr{D}^\alpha$ that will be mentioned before Lemma 3.2 in Appendix, we can see that $Mf(x) \le 6^n \sup_{\alpha\in\{0,1,2\}^n} \sup_{Q\in\mathscr{D}^\alpha} \Lambda_{\mathcal{S}(Q),1} f(x)$, where $\mathcal{S}(Q) = \{Q\}$ is the family consisting of only $Q$; see Appendix for the definition of $\mathscr{D}^\alpha$. The family $\mathcal{S}(Q)$ is obviously a sparse family. Like sparse forms $\Lambda_{\mathcal{S};r,s}$, this operator also admit universal ones, see Lemma , i.e. there exist $3^n$ sparse families $\mathcal{S}^\alpha$, $\alpha \in \{0,1,2\}^n$, depending on $f$, such that

$$\sup_{\alpha\in\{0,1,2\}^n} \sup_{Q\in\mathscr{D}^\alpha} \Lambda_{\mathcal{S}(Q),1} f \lesssim \sum_{\alpha\in\{0,1,2\}^n} \Lambda_{\mathcal{S}^\alpha;1} f.$$

Therefore, we have

$$\begin{aligned}
\left|\langle T^2_\ell f, g\rangle\right| &\lesssim 2^{-\ell N}\langle Mf, |g|\rangle \\
&\lesssim 2^{-\ell N} \sum_{\alpha\in\{0,1,2\}^n} \langle \Lambda_{\mathcal{S}^\alpha;1} f, |g|\rangle \\
&= 2^{-\ell N} \sum_{\alpha\in\{0,1,2\}^n} \Lambda_{\mathcal{S}^\alpha;1,1}(f,g).
\end{aligned}$$

## § 2.2. Sparse bound for $T^1_\ell$

Let $Q \subset \mathbb{R}^n$ be a cube satisfying supp $f$, supp $g \subset Q$ and $\ell(Q) \ge 10r$, $r := 2^{\ell(1+\varepsilon/n)}$, where $\ell(Q)$ means the side length of $Q$. We repeat dyadic decomposition of $Q$ until the side length of the subcubes is equivalent to $r$. That is $Q = \bigcup_{j=1}^K Q_j$ with $\ell(Q_j) \approx r$ and $Q_j \cap Q_k = \varnothing$ if $j \ne k$. We remark that $K \in \mathbb{N}$ depends on $\ell$. With the subcubes, we write

$$f = \sum_{j=1}^K f\chi_{Q_j} =: \sum_j f_j \quad \text{and} \quad g = \sum_{k=1}^K g\chi_{Q_k} =: \sum_k g_k,$$

then divide

$$\left|\langle T^1_\ell f, g\rangle\right| \le \sum_k \sum_{\substack{j \\ Q_j\subset 3Q_k}} \left|\langle T^1_\ell f_j, g_k\rangle\right| + \sum_k \sum_{\substack{j \\ Q_j\not\subset 3Q_k}} \left|\langle T^1_\ell f_j, g_k\rangle\right| =: I + II.$$

The first term is the resonance part and the other is regarded as an error, consisting of tail integrals.

### 2.2.1. Sparse bound for I

From Cauchy-Schwarz inequality, it follows

$$I \le \sum_k \sum_{\substack{j \\ Q_j \subset 3Q_k}} \|T_\ell^1 f_j\|_{L^2} \|g_k\|_{L^2},$$

and Hausdorff-Young inequality gives

$$\|T_\ell^1 f_j\|_{L^2} \le 2^{-\ell\delta} \|f_j\|_{L^1} \left\| \sup_{1 \le R \le 2} \left|K_{\ell,R}^1\right| \right\|_{L^2}.$$

One obtains from Plancherel's theorem

$$\begin{aligned} \left\| \sup_{1 \le R \le 2} \left|K_{\ell,R}^1\right| \right\|_{L^2} &\le \int_{\mathbb{R}} |\rho(t)| \left\| e^{i2^\ell t|\cdot|} \varphi\psi \right\|_{L^2} dt \\ &\lesssim |\mathrm{supp}\ \psi|^{1/2} \\ &\approx 2^{-\ell\tau/2}, \end{aligned}$$

thus $\|T_\ell^1 f_j\|_{L^2} \lesssim 2^{-\ell(\delta+\tau/2)} \|f_j\|_{L^1}$. Here, let $\mathcal{S} := \{3Q_k\}_{k=1}^K$ and $E_{3Q_k} := Q_k$. Obviously, this is a $3^{-n}$-sparse family. From the bound of $T_\ell^1$ above, the desired sparse bound follows:

$$\begin{aligned} I &\lesssim 2^{-\ell(\delta+\tau/2)} \sum_k \sum_{\substack{j \\ Q_j \subset 3Q_k}} \|f_j\|_{L^1} \|g_k\|_{L^2} \\ &\approx 2^{-\ell(\delta+\tau/2-(n+\varepsilon)/2)} \sum_k |3Q_k| \langle f \rangle_{3Q_k,1} \langle g \rangle_{3Q_k,2} \\ &= 2^{-\ell(\delta-(n-1)/2-\varepsilon)} \Lambda_{\mathcal{S};1,2}(f,g). \end{aligned}$$

### 2.2.2. Sparse bound for II

Following Lacey, Mena and Reguera [16], we use a variant of sparse form; for $m_1, m_2 > 0$

$$\Lambda_{\mathcal{S};r,s}^{(m_1,m_2)}(f,g) := \sum_{Q \in \mathcal{S}} |Q| \langle\!\langle f \rangle\!\rangle_{Q,r}^{(m_1)} \langle\!\langle g \rangle\!\rangle_{Q,s}^{(m_2)},$$

where

$$\langle\!\langle h \rangle\!\rangle_{Q,r}^{(m)} := \left( |Q|^{-1} \int |h(x)|^r \left(1 + \frac{\mathrm{dist}(x,Q)}{\ell(Q)}\right)^{-m} dx \right)^{1/r}.$$

This form can be controlled by standard ones, see Appendix.

It is not hard to see that $|K_{\ell,R}^1(x)| \lesssim 2^{-\ell\tau} \left( \frac{2^\ell}{\langle x \rangle} \right)^N$ for any $N \ge 0$. This is done by integration by parts for the inner integral in the definition of $K_{\ell,R}^1$ and $|\mathrm{supp}\psi| \approx 2^{-\ell\tau}$.

Thus it follows

$$II \lesssim 2^{-\ell(\delta+\tau-N)} \sum_k \sum_{\substack{j \\ Q_j \notin 3Q_k}} \int |f_j(y)| \int |g_k(x)||x-y|^{-N} dxdy.$$

We fix $k$ and denote $L^{(k)} := \{Q_j; Q_j \notin 3Q_k\}$, and we divide $L^{(k)}$ into subclasses

$$\begin{cases} L_1^{(k)} := \varnothing \quad \text{and} \\ L_\sigma^{(k)} := \{Q_j \in L^{(k)}; ((2\sigma+1)Q_k \backslash (2\sigma-1)Q_k) \cap Q_j \neq \varnothing\}, \ (\sigma = 2, 3, \cdots). \end{cases}$$

Cardinal number of $L_\sigma^{(k)}$ is smaller than a constant times $\sigma^{n-1}$. Remark that if $x \in Q_k$, $y \in Q_j$ and $Q_j \in L_\sigma^{(k)}$, then

$$|x-y| \approx \mathrm{dist}(x, Q_j) \approx \mathrm{dist}(y, Q_k) \approx \mathrm{dist}(Q_j, Q_k) \approx 2^{(1+\varepsilon/n)\ell}\sigma.$$

Using these, one obtains

$$\begin{aligned} II \lesssim 2^{-\ell(\delta+\tau-N+N(1+\varepsilon/n))} \sum_k \sum_\sigma \sigma^{-N/3} \sum_{\substack{j \\ Q_j \in L_\sigma^{(k)}}} \int |f_j(y)| \left(1 + \frac{\mathrm{dist}(y, Q_k)}{\ell(Q_k)}\right)^{-N/3} dy \\ \times \int |g_k(x)| \left(1 + \frac{\mathrm{dist}(x, Q_j)}{\ell(Q_j)}\right)^{-N/3} dx. \end{aligned}$$

Since for all $x \in Q_k$

$$1 + \frac{\mathrm{dist}(x, Q_k)}{\ell(Q_k)} \lesssim 1 + \frac{\mathrm{dist}(x, Q_j)}{\ell(Q_j)},$$

we see by taking sufficiently large $N, N'$

$$II \lesssim 2^{-\ell N'} \sum_k |Q_k| \langle\!\langle f \rangle\!\rangle_{Q_k,1}^{(N/3)} \langle\!\langle g \rangle\!\rangle_{Q_k,1}^{(N/3)},$$

Applying Proposition 3.1 below, we can find sparse families $\{\mathcal{S}^\alpha\}_{\alpha \in \{0,1,2\}^n}$, for which the right hand side in the last inequality is dominated by $2^{-\ell N} \sum_{\alpha \in \{0,1,2\}^n} \Lambda_{\mathcal{S}^\alpha;1,1}(f, g)$ provided that if $N > 15n/2$.

## § 3. Appendix

Here we give a proposition, which is used in the previous section.

**Proposition 3.1.** *(Culiuc, Kesler and Lacey [4]) Let $r, s \in [1, \infty]$, $\eta \in (0,1)$ and $f, g \in L_0^\infty$. If $m_1$, $m_2 > n(1 + 1/r + 1/s)$, then there exist $\eta$-sparse families $\{\mathcal{S}^\alpha\}_{\alpha \in \{0,1,2\}^n}$ such that for any $\eta_0 \in (0,1)$ and $\eta_0$-sparse family $\mathcal{S}_0$,*

$$\Lambda_{\mathcal{S}_0;r,s}^{(m_1,m_2)}(f, g) \lesssim \left(\eta_0(1-\eta)^{1/r+1/s}\right)^{-1} \sum_{\alpha \in \{0,1,2\}^n} \Lambda_{\mathcal{S}^\alpha;r,s}(f, g).$$

We give a sketch of the proof.

*Proof.* We can observe that for any $Q_0 \in \mathcal{S}_0$,

$$\langle\!\langle f\rangle\!\rangle_{Q_0,r}^{m_1} \lesssim |Q_0|^{-1/r} \sum_{j=0}^{\infty} 2^{-jm_1} \|f\|_{L^r(2^jQ_0)},$$

thus, it follows

$$\langle\!\langle f\rangle\!\rangle_{Q_0,r}^{m_1} \langle\!\langle g\rangle\!\rangle_{Q_0,s}^{m_2} \lesssim \sum_{j=0}^{\infty} 2^{-j(m_*-n(1/r+1/s))} \langle f\rangle_{2^jQ_0,r} \langle g\rangle_{2^jQ_0,s},$$

with $m_* := \min(m_1, m_2)$. For each $j$, $\mathcal{S}_j := \{2^jQ_0\}_{Q_0 \in \mathcal{S}_0}$ with $E_{2^jQ_0} := E_{Q_0}$ is a $2^{-jn}\eta_0$-sparse family. Using these families, we have

$$\Lambda_{\mathcal{S}_0;r,s}^{(m_1,m_2)}(f,g) \lesssim \sum_{j=0}^{\infty} 2^{-j(m_*-n(1/r+1/s))} \Lambda_{\mathcal{S}_j;r,s}(f,g).$$

An application of Lemma 3.2 below ends the proof. □

Next lemma, from Lacey and Mena [15], gives an universal sparse family that depends on functions and consists of cubes in the translated dyadic systems:

$$\mathscr{D}^\alpha := \left\{2^{-j}\left([0,1)^n + m + (-1)^j\alpha/3\right); j \in \mathbb{Z},\ m \in \mathbb{Z}^n\right\},\ \alpha \in \{0,1,2\}^n.$$

One of important properties for $\mathscr{D}^\alpha$ is the following: any cube $Q$ in $\mathbb{R}^n$ is approximated by a cube $R$ in $\mathscr{D} := \bigcup_\alpha \mathscr{D}^\alpha$ in the sense that $Q \subset R$ and $3\ell(Q) < \ell(R) \le 6\ell(Q)$. This property will be used in the proof below. In Hytönen, Lacey and Pérez [12], a more strict statement can be found. For the dyadic analysis, we also refer to Lerner and Nazarov [17].

**Lemma 3.2.** *(Lacey and Mena [15]) Let $r, s \in [1,\infty]$, $\eta \in (0,1)$ and $f, g \in L_0^\infty$. Then, there exist $\eta$-sparse families $\mathcal{S}^\alpha \subset \mathscr{D}^\alpha$, $(\alpha \in \{0,1,2\}^n)$, such that for any $\eta_0 \in (0,1)$ and $\eta_0$-sparse family $\mathcal{S}_0$,*

$$\Lambda_{\mathcal{S}_0;r,s}(f,g) \lesssim \left(\eta_0(1-\eta)^{1/r+1/s}\right)^{-1} \sum_{\alpha \in \{0,1,2\}^n} \Lambda_{\mathcal{S}^\alpha;r,s}(f,g)$$

For the completeness, we give a proof, along the argument in [15].

*Proof.* For simplicity, we denote $X(Q) := \langle f\rangle_{Q,r}\langle g\rangle_{Q,s}$, and take

$$\sigma \ge \left(n + 1 + \log_2 \frac{1}{1-\eta}\right)(1/r + 1/s).$$

For $\alpha \in \{0,1,2\}^n$ and $k \in \mathbb{Z}$, let us define

$$\mathcal{S}_k^\alpha := \{Q \in \mathscr{D}^\alpha; 2^{\sigma k} \le X(Q)\ \&\ \text{maximal with respect to inclusions}\}.$$

It is not hard to see the following properties:

$$Q \in \mathcal{S}_k^\alpha \Rightarrow\ 2^{\sigma k} \le X(Q) < 2^{\sigma k + n(1/r+1/s)} \tag{P1}$$

$$\textit{For each } \alpha \textit{ and } k,\ \mathcal{S}_k^\alpha \textit{ is pairwise disjoint} \tag{P2}$$

$$Q \in \mathcal{S}_k^\alpha,\ R \in \mathcal{S}_\ell^\alpha \textit{ with } k < \ell \Rightarrow Q \cap R = \varnothing \textit{ or } R \subsetneq Q \tag{P3}$$

The first follows from Hölder inequality, and the second can be seen from the maximality. Since $2^{\sigma k} < 2^{\sigma \ell} \le X(Q)$, the last holds.

Next, for each $Q \in \mathcal{S}_k^\alpha$, denote

$$\begin{aligned} C(Q) &:= \{P \in \mathcal{S}_{k+1}^\alpha; P \subsetneq Q\} \\ C_f(Q) &:= \{P \in C(Q); \langle f \rangle_{P,r} > a \langle f \rangle_{Q,r}\} \\ C_g(Q) &:= \{P \in C(Q); \langle g \rangle_{P,s} > b \langle g \rangle_{Q,s}\} \text{ and} \\ E_Q &:= Q \backslash \bigcup_{P \in C(Q)} P \end{aligned}$$

where $a = \left(\dfrac{2}{1-\eta}\right)^{1/r}$ and $b = \left(\dfrac{2}{1-\eta}\right)^{1/s}$. We can observe that

$$Q \in \mathcal{S}_k^\alpha \Rightarrow C(Q) = C_f(Q) \cup C_g(Q) \text{ and} \tag{P4}$$

$$|E_Q| \ge \eta |Q|. \tag{P5}$$

Indeed, if $P \in C(Q)$ and $P \notin C_f(Q) \cup C_g(Q)$, then it follows $\dfrac{2^{\sigma(k+1)}}{ab} < X(Q)$. From the condition on $\sigma$, it follows that $2^{\sigma k + n(1/r+1/s)} \le \dfrac{2^{\sigma(k+1)}}{ab}$. But (P1) ensures that this contradicts $Q \in \mathcal{S}_k^\alpha$. To verify (P5), we see that $P \in C_f(Q)$ means $|P| < a^{-r} |Q| \dfrac{\|f\|_{L^r(P)}^r}{\|f\|_{L^r(Q)}^r}$. Hence, taking (P2) into account, we have

$$|E_Q| \ge |Q| - \sum_{P \in C_f(Q)} |P| - \sum_{P \in C_g(Q)} |P| \ge (1 - a^{-r} - b^{-s})|Q| = \eta |Q|.$$

Finally, we claim that

$$\textit{if } \mathcal{S}^\alpha := \bigcup_{k \in \mathbb{Z}} \mathcal{S}_k^\alpha, \textit{ then } \{E_Q\}_{Q \in \mathcal{S}^\alpha} \textit{ is pairwise disjoint.} \tag{P6}$$

To prove this, we assume that $Q \in \mathcal{S}_k^\alpha,\ R \in \mathcal{S}_\ell^\alpha$ with $k \le \ell$ and $Q \cap R \ne \varnothing$. If $k = \ell$, there is nothing to show form (P2). Hence, we consider the case $k < \ell$. (P3) restricts us to the case: $R \subsetneq Q$. It is sufficient to show $E_Q \cap R = \varnothing$, and this is equivalent to $R \subset \bigcup_{P \in C(Q)} P$. If $\ell = k+1$, then $R \in \mathcal{S}_\ell^\alpha = \mathcal{S}_{k+1}^\alpha$. Since $R \subsetneq Q$, it follows $R \in C(Q) \subset \bigcup_{P \in C(Q)} P$. On the

other hand, if $\ell \geq k+2$, then $2^{\sigma(k+1)} < 2^{\sigma\ell} \leq X(Q)$. From the definition of $\mathcal{S}^{\alpha}_{k+1}$, there is $\tilde{R} \in \mathcal{S}^{\alpha}_{k+1}$ so that $R \subset \tilde{R}$. Taking $Q \cap \tilde{R} \neq \varnothing$ and $k < \ell$ into account, one obtains $\tilde{R} \subsetneq Q$, which means $\tilde{R} \in C(Q)$, and ends the proof of this claim.
Here we remark that (P5) and (P6) mean $\mathcal{S}^{\alpha} \subset \mathscr{D}^{\alpha}$ is an $\eta$-sparse family.

From the approximation property of $\mathscr{D}^{\alpha}$ mentioned before Lemma 3.2, for any $Q_0 \in \mathcal{S}_0$ there is $\tilde{Q}_0 \in \bigcup_{\alpha} \mathscr{D}^{\alpha}$ so that $Q_0 \subset \tilde{Q}_0$ and $3 \leq \ell(Q_0)/\ell(\tilde{Q}_0) \leq 6$. If $X(Q_0) \neq 0$, then $2^{\sigma k} \leq X(\tilde{Q}_0) < 2^{\sigma(k+1)}$ for some $k \in \mathbb{Z}$. This ensures the existence of $Q_0^* \in \mathcal{S}^{\alpha}_k$ such that $Q_0 \subset \tilde{Q}_0 \subset Q_0^*$. We use the inequality $X(Q_0) \lesssim (1-\eta)^{-(1/r+1/s)} X(Q_0^*)$ to close the proof. The desired inequality is verified as follows:

$$\begin{aligned}
\Lambda_{\mathcal{S}_0;r,s}(f,g) &= \sum_{\substack{Q_0 \in \mathcal{S}_0 \\ X(Q_0)\neq 0}} |Q_0| X(Q_0) \\
&\leq \sum_{\alpha} \sum_{Q \in S^{\alpha}} \sum_{\substack{Q_0 \in \mathcal{S}_0 \\ X(Q_0)\neq 0 \\ Q_0^* = Q}} |Q_0| X(Q_0) \\
&\lesssim (1-\eta)^{-(1/r+1/s)} \eta_0^{-1} \sum_{\alpha} \sum_{Q \in S^{\alpha}} X(Q) \sum_{\substack{Q_0 \in \mathcal{S}_0 \\ X(Q_0)\neq 0 \\ Q_0^* = Q}} |E_{Q_0}| \\
&\leq (1-\eta)^{-(1/r+1/s)} \eta_0^{-1} \sum_{\alpha} \sum_{Q \in S^{\alpha}} |Q| X(Q) \\
&\leq (1-\eta)^{-(1/r+1/s)} \eta_0^{-1} \sum_{\alpha} \Lambda_{\mathcal{S}^{\alpha};r,s}(f,g).
\end{aligned}$$

□

## Acknowledgment

The author would like to thank the referee for variable comments and suggestions. This work was partially supported by JSPS, through Grand-in-Aid for Young Scientists (B) 15K20919.

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